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NEW GLOBAL WELL-POSEDNESS RESULT FOR A HIGHER ORDER SCHRÖDINGER EQUATION

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Abstract

We establish new global results for a higher order Schrödinger equation. We show that solutions of the equation can be extended globally in Sobolev spaces of order $s > 3/5$, the method was introduced by Bourgain [1] and used in several models.

1. Introduction

In this paper we will describe global well-posedness for solutions of the initial value problem (IVP)

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$$\begin{cases} \partial_t u + ia\partial_x^2 u + b\partial_x^3 u - e|u|^2\partial_x u + eu^2\partial_x \overline{u} = 0, \; x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.1}$$

where u is a complex valued function and a, b and e are real parameters with $be \neq 0$.

The equation above is a particular case of the IVP introduced by Hasegawa and Kodama [13, 16],

$$\begin{cases} \partial_t u + ia\partial_x^2 u + b\partial_x^3 u + i\gamma|u|^2 u + d|u|^2\partial_x u + eu^2\partial_x \overline{u} = 0, \; x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \tag{1.2}$$

to study the propagation of pulses where higher dispersive effects than those one of the Schrödinger equation were taking in consideration.

Notice that depending of the values of the parameters a, b, γ, d, and e, the equation in (1.2) reduces to either the cubic Schrödinger equation

$$i\partial_t u + \partial_x^2 u + c|u|^2 u = 0,$$

or the derivative nonlinear Schrödinger equation

$$\partial_t u + i\partial_x^2 u + d|u|^2\partial_x u + eu^2\partial_x \overline{u} = 0,$$

or the complex modified Kortweg-de Vries equation

$$\partial_t u + \partial_x^3 u + d|u|^2\partial_x u = 0.$$

These equations are well known nonlinear dispersive equations and have been broadly studied in recent years.

Let $c = (d - e)a/3b$ and $u(x, t)$ be a solution of (1.2). Define

$$v(x, t) = \exp\left(i\frac{a}{3b}x + i\frac{a^3}{27b^2}t\right)u\left(x + \frac{a^2}{3b}t, t\right), \tag{1.3}$$

then $v(x, t)$ is a solution of the equation

$$\partial_t v + b\partial_x^3 v + d|v|^2\partial_x v + ev^2\partial_x \overline{v} = 0. \tag{1.4}$$

So the transformation (1.3) takes solutions of the equation in (1.2) in solutions to a complex modified Korteweg-de Vries type equation.

Regarding the study of well-posedness for the IVP (1.2), several works have been devoted to this subject, we can mention the works of Laurey [17], Staffilani [22], and Carvajal [2]. In best local result obtained so far is due to Staffilani [22]. She proved that for data in $H^s(\mathbb{R})$, $s \geq 1/4$, the IVP (1.2) is locally well-posed.

In [17] it was shown that the flow associated to the IVP (1.2) leaves invariant on time the following quantities

$$J_1(v) = \int_{\mathbb{R}} |v|^2(x, t)dx,$$

$$J_2(v) = c_1 \int_{\mathbb{R}} |\partial_x v|^2(x, t)dx + c_2 \int_{\mathbb{R}} |v|^4(x, t)dx$$

$$+ c_3 \operatorname{Im} \int_{\mathbb{R}} v(x, t)\partial_x \overline{v(x, t)}dx,$$

where $c_1 = 3be \neq 0$, $c_2 = -e(e + d)/2$ and $c_3 = 3bc - a(e + d)$. These quantities were used in [17] to establish global well-posedness for (1.2) in $H^s(\mathbb{R})$, $s \geq 1$.

Recently when $be \neq 0$ for the IVP (1.2), Wang [26] proved global well-posedness in $H^s(\mathbb{R})$ for $s > 6/7$. Carvajal using techniques of Colliander et al. [6, 7, 8] (almost conserved quantities and I-method as in [8, 26]) obtained in [3] a sharp global well-posedness result in $H^s(\mathbb{R})$ for $s > 1/4$ under the condition

$$\gamma = a(d - e)/(3b), \; be \neq 0. \tag{1.5}$$

Here we prove the global well-posedness for the IVP (1.1) in $H^s(R)$, $s > 3/5$, using Bourgain's technique as in [9]. Observe that if $a \neq 0$, then (1.1) does not satisfy the condition (1.5) because that in this case is $\gamma = 0$ and $d = -e \neq 0$. Therefore our result is new in the literature and better that the global well-posedness result obtained in [26].

Note that (1.1) conserves the following quantities:

$$I_1(v) = \int_{\mathbb{R}} |\, v\,|^2(x,\, t)dx,\ I_2(v) = \int_{\mathbb{R}} |\, \partial_x v\,|^2(x,\, t)dx. \tag{1.6}$$

We do not know whether or not the approach in [6, 7, 8] can be applied in this case to go all the way to $s \geq 1/4$.

The local result we will use is due to Staffilani [22], it reads as the following.

Theorem 1.1. *Let* $u_0 \in H^s(\mathbb{R})$, $s \geq 1/4$, *and* $a, b \in \mathbb{R}$, $b \neq 0$, $c, d, e \in \mathbb{C}$, *then there exist*

$$\Delta T \leq C \min\{\| u_0 \|_{H^{1/4}}^{-4}, \| u_0 \|_{H^{1/4}}^{-8/3}\} \tag{1.7}$$

and a unique solution of the IVP (1.2), *such that*

$$u \in \mathcal{C}([0,\, \Delta T];\, H^s(R)), \tag{1.8}$$

$$\| \partial_x u \|_{L_x^\infty L_{\Delta T}^2} + \| D_x^s \partial_x u \|_{L_x^\infty L_{\Delta T}^2} < \infty, \tag{1.9}$$

$$\| D_x^{s-1/4} \partial_x u \|_{L_x^{20} L_{\Delta T}^{5/2}} < \infty, \tag{1.10}$$

$$\| u \|_{L_x^5 L_{\Delta T}^{10}} + \| D_x^s u \|_{L_x^5 L_{\Delta T}^{10}} < \infty, \tag{1.11}$$

$$\| u \|_{L_x^4 L_{\Delta T}^\infty} < \infty, \tag{1.12}$$

$$\| u \|_{L_x^8 L_{\Delta T}^8} + \| D_x^s u \|_{L_x^8 L_{\Delta T}^8} < \infty. \tag{1.13}$$

Moreover, for any $T' \in [0, \Delta T]$ *there exists a neighborhood* $\mathcal{V}$ of $u_0 \in H^s(\mathbb{R})$, $s \geq 1/4$, *such that the map* $\tilde{u}_0 \to \tilde{u}(t)$, *from* $\mathcal{V}$ *into the class defined by* (1.8)-(1.13), *with* T' *instead of* ΔT *is smooth.*

Our aim in this paper is to extend the local solution from the Theorem 1.1 to a global one. Now, we state our main theorem of global existence:

Theorem 1.2. *Let* $s > 3/5$, *then for any* $u_0 \in H^s(\mathbb{R})$ *the unique solution of the IVP* (1.2) *given by Theorem* 1.1 *extends to any interval* $[0, T]$. *In addition,*

$$\| u \|_{L^\infty_{[0,T]}\dot{H}^\theta} \leq c_1 T^{2\theta(1-s)/(5s-3)} + \frac{\| u_0 \|_{\dot{H}^s}}{T^{2(s-\theta)/(5s-3)}},$$

where $c_1 = c_1(\| u_0 \|_{H^s})$ *and* $0 \leq \theta \leq s$, *moreover*

$$u(t) = U(t)u_0 + \omega(t),$$

where $U(t)$ *is the unitary group associated with the linear part of* (1.1),

$$\| w \|_{L^\infty_{[0,T]}\dot{H}^\delta} \leq c_1 T^{2\delta(1-s)/(5s-3)},$$

where $c_1 = c_1(\| u_0 \|_{H^s})$, $s \leq \delta \leq 1$.

Notation. The notation to be used is mostly standard. We will use the space-time Lebesgue space $L^p_x L^q_T$ endowed with the norm

$$\| f \|_{L^p_x L^q_T} = \Big\| \| f \|_{L^q_T} \Big\|_{L^p_x} = \left(\int_{\mathbb{R}} \left(\int_0^T | f(x, t) |^q \, dt \right)^{p/q} dx \right)^{1/p}.$$

The notation $A \lesssim B$ means there exist a constant C such that $A \leq CB$, and $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2. Linear Estimates

In this section we will present the linear estimates we need to obtain our global results. Some of these linear estimates were already established somewhere else and so we will give the references.

We consider the linear problem

$$\begin{cases} \partial_t u + ia\partial_x^2 u + b\partial_x^3 u = 0, \; x, t \in \mathbb{R}, \\ u(x, 0) = u_0(x), \end{cases} \tag{2.1}$$

whose solution is given by the unitary group $\{U(t)\}$ defined via the Fourier transform as

$$U(t)u_0(x) = \int_{\mathbb{R}} e^{i(a\xi^2 + b\xi^3)t + ix\xi} \hat{u}_0(\xi) d\xi. \tag{2.2}$$

For $u_0(x)$ in $H^s(\mathbb{R})$, $s \in [1/4, 1)$, we write

$$u_0(x) = (\chi_{\{|\xi|<N\}}\hat{u}_0)^{\vee}(x) + (\chi_{\{|\xi|\geq N\}}\hat{u}_0)^{\vee}(x) = v_0(x) + w_0(x), \tag{2.3}$$

where N is a large number to be chosen later. We notice that by the definition (2.3) the both data $v_0 \in H^{\infty} := \bigcap_{j=1}^{\infty} H^j$ and $w_0 \in H^s$, satisfy

$$\| v_0 \|_{L^2} \leq \| u_0 \|_{L^2}$$

$$\| v_0 \|_{H^\delta} \leq \| u_0 \|_{H^s} N^{\delta(1-s)}, \quad \delta \geq s, \tag{2.4}$$

and

$$\| w_0 \|_{H^\rho} \leq \| u_0 \|_{H^s} N^{(\rho-s)}, \; 0 \leq \rho \leq s. \tag{2.5}$$

Then to v_0 and w_0 we associate the IVP's

$$\begin{cases} \partial_t v + ia\partial_x^2 v + b\partial_x^3 v + F(v) = 0, \; x, \, t \in \mathbb{R}, \\ v(x, 0) = v_0(x), \end{cases} \tag{2.6}$$

and

$$\begin{cases} \partial_t w + ia\partial_x^2 w + b\partial_x^3 w + G(v, w) = 0, \; x, \, t \in \mathbb{R}, \\ w(x, 0) = w_0(x), \end{cases} \tag{2.7}$$

where $F(g) = -e| g |^2 \partial_x g + eg^2 \partial_x \overline{g}$, and

$$\begin{aligned} G(v, w) := F(v + w) - F(v) = -e[| v |^2 \partial_x w + 2\partial_x v \Re e(v\overline{w}) + 2\partial_x w \Re e(v\overline{w}) \\ + | w |^2 \partial_x v + | w |^2 \partial_x w] + e[2vw\partial_x \overline{v} + 2vw\partial_x \overline{w} + v^2 \partial_x \overline{w} \\ + w^2 \partial_x \overline{v} + w^2 \partial_x \overline{w}]. \end{aligned} \tag{2.8}$$

The solution of IVP (2.6) is given by

$$v(t) = U(t)v_0 - \int_0^t U(t - t')F(v)(t')dt' \tag{2.9}$$

with $v \in H^{\infty}$ (see Theorem 1.1). This solution is defined for any given time $T > 0$, in particular it is defined in the interval of time $[0, \Delta T]$.

Proposition 2.1. *Let* $w_0 \in H^{\rho}$, *with* $\| w_0 \|_{H^{\rho}} \lesssim N^{\rho - s}$, $1/4 \leq \rho \leq s < 1$, $s \leq \delta$ *and* v *the unique solution given by Theorem* 1.1. *Then there exists a unique solution* w *of the IVP* (2.7) *defined in the same interval of existence of* v, $[0, \Delta T]$, $\Delta T \sim N^{-(1-s)}$ *such that*

$$w \in \mathcal{C}([0, \Delta T];\ H^{\rho}(\mathbb{R})).$$

Proof. The proof follows the same argument as in [15], see [9].

The Theorem (2.1) shows that the IVP 2.7 is locally well-posed in H^{ρ} and it is given by

$$\begin{aligned} w(t) &= U(t)w_0(x) - \int_0^t U(t-t')G(v, w)(t')dt' \\ &= U(t)w_0(x) + z(t) \end{aligned} \tag{2.10}$$

in the interval of time $[0, \Delta T]$. Therefore,

$$u(t) = v(t) + z(t) + U(t)w_0,\ t \in [0, \Delta T]. \tag{2.11}$$

The following lemma establishes the smoothing effect of Kato's type and its dual version and it allows us to estimate the nonlinear term without using Leibnitz rule for fractional derivatives as in [9], [10].

Lemma 2.2. *If* $u_0 \in L^2(\mathbb{R})$, $D^{\theta}v_0 \in L^2(\mathbb{R})$, $0 < \theta \leq 1$. *Then*

$$\| \partial_x U(t)u_0(x) \|_{L_x^{\infty}L_T^2} \leq C\| u_0 \|_{L^2}, \tag{2.12}$$

$$\| \partial_x U(t)v_0(x) \|_{L_x^{2/\theta}L_T^2} \leq CT^{\theta/2}\| D^{\theta}v_0 \|_{L^2}. \tag{2.13}$$

Proof. To the proof of (2.12), see [22] or [2], and (2.13) follows using complex interpolation between (2.12) and the inequality

$$\| U(t)u_0(x) \|_{L_x^2L_T^2} \leq T^{1/2}\| u_0 \|_{L^2}. \tag{2.14}$$

Now, using Lemma 2.2 we obtain some interpolated estimates.

Lemma 2.3. *Let* $0 < \theta \leq 1$. *If* $f \in L_x^{2/(1+\theta)}L_T^2$. *Then*

$$\left\| D_x^{\theta}\int_0^t U(t-t')f(\cdot,t')dt' \right\|_{L_T^{\infty}L_x^2} \le CT^{(1-\theta)/2}\| f \|_{L_x^{2/(1+\theta)}L_T^2}. \tag{2.15}$$

Proof. The Minkowski inequality, group properties and the Cauchy-Schwarz inequality give

$$\left\| \int_0^t U(t-t')f(\cdot,t')dt' \right\|_{L_x^2} \le CT^{1/2}\| f \|_{L_x^2L_T^2}.$$

This estimate and version dual of (2.12) combined with Stein's analytic interpolation give us (2.15).

Next we have the estimates associated to the maximal function norm.

Lemma 2.4. *If* $u_0 \in H^s$, $s > 3/4$, $0 < T < 1$, *then*

$$\| U(t)u_0 \|_{L_x^2L_T^{\infty}} \le C\| u_0 \|_{H^s}. \tag{2.16}$$

If $u_0 \in H^{1/4}$, *then*

$$\| U(t)u_0 \|_{L_x^4L_T^{\infty}} \le C\| u_0 \|_{H^{1/4}}. \tag{2.17}$$

If $u_0 \in H^{(1+2\theta^+)/4}$, $0 \le \theta \le 1$ *and* $0 < T < 1$, *then*

$$\| U(t)u_0 \|_{L_x^{4/(1+\theta)}L_T^{\infty}} \le C\| u_0 \|_{H^{(1+2\theta^+)/4}}. \tag{2.18}$$

Proof. The estimate (2.16) was proved in [17]. Inequality (2.17) was obtained by Staffilani in [22], (see also [2]). To show (2.18) we use (2.16), (2.17) and analytic interpolation.

We also need to establish Strichartz estimates for solutions of IVP (1.2). The following lemma was proved in [2]

Lemma 2.5. *If* $u_0 \in L^2$, *then*

$$\| U(t)u_0 \|_{L_x^8L_T^8} \le C\| u_0 \|_{L_x^2}. \tag{2.19}$$

We also need the estimates established in [22] reminiscence of those for the modified KdV equation.

Lemma 2.6. *If* u_0, $D_x^{1/4}\tilde{u}_0 \in L^2$, *then*

$$\| U(t)u_0 \|_{L_x^5 L_T^{10}} \leq C\| u_0 \|_{L_x^2}, \tag{2.20}$$

$$\| \partial_x U(t)\widetilde{u}_0 \|_{L_x^{20} L_T^{5/2}} \leq C\| D_x^{1/4}\widetilde{u}_0 \|_{L_x^2}. \tag{2.21}$$

Now, we state a result of interpolation which allows us to handle the term vv_xw.

Lemma 2.7. *Let* $1/4 \leq \theta \leq 1$, $D^\theta u_0 \in L^2$, *then*

$$\| D_x U(t)u_0 \|_{L_x^{40/(20\theta-3)} L_T^{5/2}} \leq CT^{\theta/2-1/8}\| D^\theta u_0 \|_{L_x^2}. \tag{2.22}$$

Proof. Interpolating the effect of local regularization (2.12) with the maximal function estimative (2.17), for $0 \leq \theta \leq 1$ we have that

$$\| D_x U(t)u_0 \|_{L_x^{5/\theta} L_T^{10/(5-4\theta)}} \leq C\| D^\theta u_0 \|_{L_x^2}. \tag{2.23}$$

Now, we interpolate (2.23) with (2.13) to obtain (2.22), because if $\lambda = 1 - 1/(4\theta)$ we have

$$\frac{20\theta - 3}{40} = \lambda\frac{\theta}{2} + (1-\lambda)\frac{\theta}{5}.$$

$$\frac{2}{5} = \lambda\frac{1}{2} + (1-\lambda)\frac{5-4\theta}{10}.$$

This completes the proof.

To prove Theorem 1.2 we need the following result

3. Estimatives Norms

Notice that we have added the norms in (1.13) to the original set of norms in [22] to avoid further difficulties. In what follows we consider N as a large enough positive number, which will be chosen later. We define

$$h^\beta(\phi) := \| \phi \|_{L_{\Delta T}^\infty H^\beta} + \|D_x^\beta \partial_x \phi\|_{L_x^\infty L_{\Delta T}^2} + \| \partial_x\phi \|_{L_x^\infty L_{\Delta T}^2} + \| \phi \|_{L_x^2 L_{\Delta T}^\infty}$$

$$+ \|D_x^\beta \phi\|_{L_x^5 L_{\Delta T}^{10}} + \| \phi \|_{L_x^5 L_{\Delta T}^{10}} + \|D_x^{\beta-1/4}\partial_x\phi\|_{L_x^{20} L_{\Delta T}^{5/2}} + \| \phi \|_{L_x^8 L_T^8}$$

$$+\|D_x^{\beta}\phi\|_{L_x^8L_T^8},$$

$$g(\phi) := \|\phi\|_{L^\infty_{\Delta T}H^{1/4}} + \|D_x^{1/4}\partial_x\phi\|_{L_x^\infty L^2_{\Delta T}} + \|\partial_x\phi\|_{L_x^{20}L^{5/2}_{\Delta T}}$$

$$+\|\phi\|_{L_x^4L^\infty_{\Delta T}} + \|D_x^{1/4}\phi\|_{L_x^5L^{10}_{\Delta T}} + \|\phi\|_{L_x^5L^{10}_{\Delta T}}$$

$$+\|\partial_x\phi\|_{L_x^\infty L^2_{\Delta T}} + \|\phi\|_{L_x^8L^8_{\Delta T}} + \|D_x^{1/4}\phi\|_{L_x^8L^8_{\Delta T}},$$

$$f(\phi) := \|\phi\|_{L^\infty_{\Delta T}L^2_x} + \|\partial_x\phi\|_{L_x^\infty L^2_{\Delta T}} + \|\phi\|_{L_x^8L^8_{\Delta T}} + \|\phi\|_{L_x^5L^{10}_{\Delta T}}.$$

Lemma 3.1. *Suppose that v and w solutions of the IVP's* (2.6) *and* (2.7) *with initial data w_0, v_0 respectively. Let* $1/4 \le \rho \le s < 1$, $\delta \in [0, 1]$, $|||w|||_\rho := h^\rho(w)$, $|||v|||_\delta := h^\delta(v)$, $\|w\|_{1/4} := g(w)$, $\|v\|_{1/4} := g(v)$, $||||w||||_0 := f(w)$, $||||v||||_0 := f(v)$, *then*

$$|||v|||_\delta \le C\|v_0\|_{H^\delta} \le CN^{\delta(1-s)},\ \|v\|_{1/4} \le C\|v_0\|_{H^{1/4}} \le CN^{\frac{1}{4}(1-s)},$$

$$||||v||||_0 \le C\|v_0\|_{L^2} \le C$$

and

$$|||w|||_\rho \le C\|w_0\|_{H^\rho} \le CN^{(\rho-s)},\ \|w\|_{1/4} \le C\|w_0\|_{H^{1/4}} \le CN^{\left(\frac{1}{4}-s\right)}$$

$$||||w||||_0 \le C\|w_0\|_{L^2} \le CN^{-s}.$$

Proof. The proof of these inequalities is the same as in [9]

Lemma 3.2. *If v is a solution of PVI* (2.6) *in* $[0, \Delta T]$, *then, for* $\theta \in [0, 1]$, $\mu \in [1/4, 1]$

$$\|\partial_x v\|_{L_x^{2/\theta}L^2_{\Delta T}} \le C\Delta T^{\theta/2}|||v|||_\theta. \tag{3.1}$$

$$\|\partial_x v\|_{L_x^{40/(20\mu-3)}L^{5/2}_{\Delta T}} \le C\Delta T^{\mu/2-1/8}|||v|||_\mu. \tag{3.2}$$

If $\Delta T < 1$,

$$\| v \|_{L_x^{4/(1+\theta)} L_{\Delta T}^{\infty}} \leq C \||\, v \,\||_{(1/4+\theta^+/2)}. \tag{3.3}$$

Moreover, we have that the solution w of PVI (2.7) *satisfies*,

$$\| \partial_x w \|_{L_x^{2/\theta} L_{\Delta T}^2} \leq C \Delta T^{\theta/2} \||\, w \,\||_{\theta}. \tag{3.4}$$

$$\| \partial_x w \|_{L_x^{40/(20\mu-3)} L_{\Delta T}^{5/2}} \leq C \Delta T^{\mu/2-1/8} \||\, w \,\||_{\mu}. \tag{3.5}$$

If $\Delta T < 1$, *one has*

$$\| w \|_{L_x^{4/(1+\theta)} L_{\Delta T}^{\infty}} \leq C \||\, w \,\||_{(1/4+\theta^+/2)}. \tag{3.6}$$

Proof. To prove (3.1) we use (2.13) and the integral formula for v (2.9). Let $a = 2C \| v_0 \|_{H^\theta}$ and ΔT such that $\Delta T^{1/2}(1 + \Delta T^{1/4}) a^2 \leq 1/2$ for $\||\, v \,\||_{\theta} \leq a$, thus we obtain (see the Theorem 1.1 in [22], see also [2])

$$\begin{aligned} \| \partial_x v \|_{L_x^{2/\theta} L_{\Delta T}^2} &\leq C \Delta T^{\theta/2} \| v_0 \|_{H^\theta} + C \Delta T^{\theta/2} \Delta T^{1/2} (1 + \Delta T^{1/4}) \||\, v \,\||_{\theta}^3 \\ &\leq C \Delta T^{\theta/2} \| v \|_{L_{\Delta T}^{\infty} H^\theta} + C \Delta T^{\theta/2} \||\, v \,\||_{\theta}. \end{aligned}$$

This proves (3.1).

The other inequalities are obtained analogously by using the inequalities (2.22) and (2.18) of Lemma 2.7 and Lemma 2.4 respectively.

Proposition 3.3. *Define*

$$z(t) = -\int_0^t U(t - t') G(w, v)(t') dt',$$

where $G(w, v)$ is the same as in (2.8), *with v and w solutions of the IVP's* (2.6) *and* (2.7) *respectively*, *with intial data v_0 and w_0 as in* (2.3). *Then for $\delta \in [0, 1/4]$, $3/5 < s < 1$ we have*

$$\begin{aligned} \|D_x^{\delta} z\|_{L_{\Delta T}^{\infty} L^2} &\lesssim N^{\left\{\frac{3\delta}{2} - \left(1 + \frac{3\delta}{2}\right) s\right\}}, \\ \| z \|_{L_{\Delta T}^{\infty} L^2} &\lesssim N^{-s}. \end{aligned} \tag{3.7}$$

Proof. The terms that contain v are more difficult to limit than those which contain w because the triple norm of v are limited by positive potencies of N, and those w by negative potencies of N. We consider the worst term, $|v|^2 w_x$, $vw\bar{v}_x$ and also $i|w|^2 w$, $w^2\overline{w_x}$. Using (2.15) and the Lemmas 3.1 and 3.2 we have

$$
\begin{aligned}
\left\| D_x^{\delta}\int_0^t U(t-t')|v|^2 w_x(t')dt' \right\|_{L^{\infty}_{\Delta T}L^2} &\leq C\Delta T^{\frac{1-\delta}{2}} \||v|^2 w_x\|_{L_x^{\frac{2}{1+\delta}}L^2_{\Delta T}} \\
&\leq C\Delta T^{\frac{1-\delta}{2}} \|v\|^2_{L_x^{\frac{4}{1+\delta}}L^{\infty}_{\Delta T}} \|w_x\|_{L^{\infty}_x L^2_{\Delta T}} \\
&\leq C\Delta T^{\frac{1-\delta}{2}} |||v|||^2_{\frac{1+2\delta}{4}} |||w|||_0 \\
&\leq CN^{-(1-s)\left(\frac{1}{2}-\frac{\delta}{2}\right)} N^{2\left(\frac{1}{4}+\frac{\delta}{2}\right)(1-s)} N^{-s} \\
&\leq CN^{\left\{\frac{3\delta}{2}-\left(1+\frac{3\delta}{2}\right)s\right\}}
\end{aligned}
$$

and

$$
\left\| D_x^{\delta}\int_0^t U(t-t')vw\bar{v}_x(t')dt' \right\|_{L^{\infty}_{\Delta T}L^2} \leq C\Delta T^{\frac{1-\delta}{2}} \|vw\bar{v}_x\|_{L_x^{\frac{2}{1+\delta}}L^2_{\Delta T}}
$$

$$
\begin{aligned}
&\leq C\Delta T^{\frac{1-\delta}{2}} \|v_x\|_{L_x^{\frac{40}{20\delta-3}}L^{5/2}_{\Delta T}} \|v\|_{L_x^{8/3}L^{\infty}_{\Delta T}} \|w\|_{L^5_x L^{10}_{\Delta T}} \\
&\leq C\Delta T^{\frac{1-\delta}{2}} |||v|||_{\delta} |||v|||_{1/2} |||w|||_0 \\
&\leq CN^{-(1-s)\left(\frac{1}{2}-\frac{\delta}{2}\right)} N^{\delta(1-s)} N^{\frac{1}{2}(1-s)} N^{-s} \\
&\leq CN^{\left\{\frac{3\delta}{2}-\left(1+\frac{3\delta}{2}\right)s\right\}}.
\end{aligned}
$$

Using (2.15), the Lemmas 3.1 and 3.2 we have

$$
\begin{aligned}
\left\| D_x^{\delta}\int_0^t U(t-t')w^2\overline{w}_x(t')dt' \right\|_{L^\infty_{\Delta T}L^2} &\le C\Delta T^{\frac{1-\delta}{2}} \|w^2\overline{w}_x\|_{L_x^{\frac{2}{1+\delta}}L^2_{\Delta T}} \\
&\le C\Delta T^{\frac{1-\delta}{2}} \| w \|^2_{L^4_x L^\infty_{\Delta T}} \| w_x \|_{L_x^{\frac{2}{\delta}}L^2_{\Delta T}} \\
&\le C\Delta T^{\frac{1-\delta}{2}}\Delta T^{\frac{\delta}{2}} \||w\||_{\delta} \| w \|^2_{1/4} \\
&\le CN^{-\frac{1}{2}\left(1-\frac{s}{\delta}\right)} N^{\frac{1}{2}\left(1-\frac{s}{\delta}\right)} N^{\delta-s} \\
&\le CN^{\delta-s}.
\end{aligned}
$$

We also have

$$
\begin{aligned}
\left\| D_x^{\delta}\int_0^t U(t-t')|w|^2 w(t')dt' \right\|_{L^\infty_{\Delta T}L^2} &\le C\Delta T^{\frac{1-\delta}{2}} \||w|^2 w\|_{L_x^{\frac{2}{1+\delta}}L^2_{\Delta T}} \\
&\le C\Delta T^{\frac{1-\delta}{2}} \| w \|_{L^4_x L^\infty_{\Delta T}} \| w \|_{L^8_x L^2_{\Delta T}} \| w \|_{L_x^{\frac{8}{1+4\delta}} L^\infty_{\Delta T}} \\
&\le C\Delta T^{\frac{7}{8}-\frac{\delta}{2}} \| w \|_{1/4} \| w \|_{L^8_x L^8_{\Delta T}} \||w\||_{\delta} \\
&\le CN^{-\left(1-\frac{s}{\delta}\right)\left(\frac{7}{8}-\frac{\delta}{2}\right)} N^{\frac{1}{4}\left(1-\frac{s}{\delta}\right)} N^{\delta-s} \\
&\le CN^{\delta-s}.
\end{aligned}
$$

Now we prove (3.7) for the worst terms

$$
\begin{aligned}
\left\| \int_0^t U(t-t')|v|^2\partial_x w dt' \right\|_{L^\infty_{\Delta T}L^2} &\le C\Delta T^{\frac{1}{2}} \||v|^2\partial_x w\|_{L^2_x L^2_{\Delta T}} \\
&\le C\Delta T^{\frac{1}{2}} \| v \|^2_{L^4_x L^\infty_{\Delta T}} \| \partial_x w \|_{L^\infty_x L^2_{\Delta T}}
\end{aligned}
$$

$$\leq CN^{-1/2(1-s)}N^{1/2(1-s)}N^{-s}$$

$$\leq CN^{-s},$$

and

$$\left\| \int_0^t U(t-t')vw\bar{v}_x dt' \right\|_{L^\infty_{\Delta T} L^2} \leq C\Delta T^{\frac{1}{2}} \| vw\bar{v}_x \|_{L^2_x L^2_{\Delta T}}$$

$$\leq C\Delta T^{\frac{1}{2}} \| v \|_{L^4_x L^\infty_{\Delta T}} \| w \|_{L^4_x L^\infty_{\Delta T}} \| \partial_x v \|_{L^\infty_x L^2_{\Delta T}}$$

$$\leq CN^{-1/2(1-s)}N^{1/4(1-s)}N^{1/4-s}$$

$$\leq CN^{-s}.$$

Remark 3.4. If $1/4 \leq s \leq \delta < 1$, then $N^{\left\{\frac{3\delta}{2} - \left(1 + \frac{3\delta}{2}\right)s\right\}} \leq N^{(1-s)\delta}$.

4. Proof of Theorem 1.2

Proof. Let $T > 0$ and $3/5 < s$, we will prove that u, solution of PVI (1.2) is globally well-posed in H^s.

To extend the local solution as far as T, we follow the scheme:

From (2.11), we have

$$u(\Delta T) = v(\Delta T) + z(\Delta T) + U(\Delta T)w_0.$$

Let $\overset{0}{v}(t) := v(t)$, $\overset{0}{z}(t) := z(t)$, $t \in [0, \Delta T]$, we defined our new initial data by

$$v_1 = \overset{0}{v}(\Delta T) + \overset{0}{z}(\Delta T) \text{ and } w_1 = U(\Delta T)w_0. \tag{4.1}$$

Obviously w_1 satisfies (2.5). By interpolation, conserved quantities (1.6) and Proposition 3.3, v_1 satisfies (2.4), therefore we can extend our local solution $u(t)$ until $2\Delta T$ and for $t \in [0, \Delta T]$ we have

$$
\begin{aligned}
u(t+\Delta T) &= \overset{1}{v}(t) + \overset{1}{w}(t) \\
&= \overset{1}{v}(t) + \overset{1}{z}(t) + U(t)w_1, \qquad (4.2)
\end{aligned}
$$

where $\overset{1}{v}$ and $\overset{1}{w}$ are solutions of the IVP (2.6), (2.7) with initial dates v_1 and w_1, respectively.

In time $2\Delta T$ repeat the process. We define by induction

$$v_{k+1} = \overset{k}{v}(\Delta T) + \overset{k}{z}(\Delta T),\ w_{k+1} = U(\Delta T)w_k = U((k+1)\Delta T)w_0, \qquad (4.3)$$

where $\overset{k}{v}(t)$, $\overset{k}{w}(t)$, $t \in [0, \Delta T]$ satisfy the IVP (2.6), (2.7) with initial dates v_k and w_k, respectively, and for $t \in [0, \Delta T]$

$$\overset{k}{z}(t) = -\int_0^t U(t-t')G(\overset{k}{v}, \overset{k}{w})(t')dt'. \qquad (4.4)$$

We will prove that for $k = 1, 2, \ldots, n$, $n \approx T/\Delta T$

$$\| v_0 \|_{L^2} + \sum_{j=0}^{k-1} \|\overset{j}{z}\|_{L^\infty_{\Delta T}L^2} \le \frac{c_0}{2} + kcc_0^2 N^{-s} \le c_0, \qquad (4.5)$$

$$\| v_0 \|_{\dot{H}^1} + \sum_{j=0}^{k-1} \|\overset{j}{z}\|_{L^\infty_{\Delta T}\dot{H}^1} \le \frac{c_0}{2} N^{(1-s)} + kcc_0^2 N^{\frac{3}{2}-\frac{5}{2}s} \le c_0 N^{(1-s)}, \qquad (4.6)$$

and

$$\| v_k \|_{\dot{H}^\delta} \le c_0 N^{\delta(1-s)},\ \delta \in [0, 1], \qquad (4.7)$$

where $c_0 = 2\| u_0 \|_{H^s}$. Observe that in the inequalities (4.5) and (4.6), the second inequality is true. In fact we have

$$kcc_0^2 N^{\frac{3}{2}-\frac{5}{2}s} \le \left(\frac{T}{\Delta T}\right) cc_0^2 N^{\frac{3}{2}-\frac{5}{2}s},$$

then the second inequality in (4.6) is true if

$$\left(\frac{T}{\Delta T}\right)cc_0^2N^{\frac{3}{2}-\frac{5}{2}s} = \frac{c_0}{2}N^{(1-s)},$$

this implies

$$T = \frac{1}{2cc_0}N^{-\frac{3}{2}+\frac{5}{2}s}, \text{ and } s > 3/5.$$

In the second inequality in (4.5) we have

$$kcc_0^2N^{-s} \le \left(\frac{T}{\Delta T}\right)cc_0^2N^{-s} \le \frac{1}{2cc_0}N^{-\frac{3}{2}+\frac{5}{2}s}N^{1-s}cc_0^2N^{-s} = \frac{c_0}{2}N^{\frac{1}{2}s-\frac{1}{2}}$$

$$\le \frac{c_0}{2},$$

therefore the second inequality in (4.5) is also true.

Now we will prove the inequalities (4.5)-(4.7) using induction. We consider the worst term, $vw\bar{v}_x$, and thus $G(v, w) = vw\bar{v}_x$.

(1) If $k = 1$

$$v_1 = \overset{0}{v}(t) + \overset{0}{z}(t).$$

The inequality (2.4) and Proposition 3.3 imply

$$\| v_0 \|_{L^2} + \|\overset{0}{z}\|_{L^\infty_{\Delta T}L^2} \le \frac{c_0}{2} + cc_0^2N^{-s} \le c_0. \tag{4.8}$$

$$\| v_0 \|_{\dot{H}^1} + \|\overset{0}{z}\|_{L^\infty_{\Delta T}\dot{H}^1} \le \frac{c_0}{2}N^{(1-s)} + cc_0^2N^{\frac{3}{2}-\frac{5}{2}s} \le c_0N^{(1-s)}, \tag{4.9}$$

by interpolation, definition of v_1 and conserved quantities (1.6) we get

$$\| v_1 \|_{\dot{H}^\delta} \le \| v_1 \|_{L^2}^{1-\delta} \| v_1 \|_{\dot{H}^1}^{\delta}$$

$$\le (\| v_0 \|_{L^2} + \|\overset{0}{z}(\Delta T)\|_{L^2})^{1-\delta}(\| v_0 \|_{\dot{H}^1} + \|\overset{0}{z}(\Delta T)\|_{\dot{H}^1})^{\delta}.$$

The inequalities (4.8) and (4.9) give

$$\| v_1 \|_{\dot{H}^\delta} \le c_0^{1-\delta} c_0^\delta N^{\delta(1-s)} \le c_0 N^{\delta(1-s)}.$$

(2) We suppose that the inequalities (4.5)-(4.7) are true for k, we will prove these inequalities for $k+1$.

Let $\zeta \in \{0, 1\}$, by (4.4), Proposition 3.3, Lemma 3.1 and inequality (4.7), we obtain

$$\begin{aligned}
\|\overset{k}{z}\|_{L^\infty_{\Delta T}\dot{H}^\zeta} &\le c\Delta T^{\frac{1-\zeta}{2}} \|\overset{k}{v}_x\|_{L_x^{\frac{40}{20\zeta-3}} L_{\Delta T}^{5/2}} \|\overset{k}{v}\|_{L_x^{8/3} L_{\Delta T}^\infty} \|\overset{k}{w}\|_{L_x^5 L_{\Delta T}^{10}} \\
&\le c\Delta T^{(1-\zeta)/2} \| v_{k-1} \|_{\dot{H}^\zeta} \| v_{k-1} \|_{\dot{H}^{1/2}} \| w_{k-1} \|_{L^2} \\
&\le c c_0^2 \Delta T^{(1-\zeta)/2} N^{\zeta(1-s)} N^{1/2(1-s)} N^{-s} \\
&\le c c_0^2 N^{\frac{3\zeta}{2} - \left(1 + \frac{3\zeta}{2}\right)s}. \qquad (4.10)
\end{aligned}$$

Using (4.3), conserved quantities (1.6), the inequalities (4.5), (4.6) and (4.10) we have

$$\begin{aligned}
\| v_{k+1} \|_{\dot{H}^\zeta} &\le \| v_0 \|_{\dot{H}^\zeta} + \sum_{j=0}^{k} \|\overset{j}{z}(\Delta T)\|_{\dot{H}^\zeta} \\
&\le \frac{c_0}{2} + k c c_0^2 N^{\frac{3\zeta}{2} - \left(1 + \frac{3\zeta}{2}\right)s} + c c_0^2 N^{\frac{3\zeta}{2} - \left(1 + \frac{3\zeta}{2}\right)s} \\
&\le \frac{c_0}{2} + (k+1) c c_0^2 N^{\frac{3\zeta}{2} - \left(1 + \frac{3\zeta}{2}\right)s} \\
&\le c_0 N^{\zeta(1-s)}, \ \zeta = 0, 1. \qquad (4.11)
\end{aligned}$$

Therefore using interpolation and (4.11)

$$\begin{aligned}
\| v_{k+1} \|_{\dot{H}^\delta} &\le \| v_{k+1} \|_{L^2}^{1-\delta} \| v_{k+1} \|_{\dot{H}^1}^{\delta} \\
&\le c_0^{1-\delta} c_0^\delta N^{\delta(1-s)} \\
&\le c_0 N^{\delta(1-\delta)}, \ \delta \in [0, 1].
\end{aligned}$$

Hence u is globally well-posed in H^s for all $3/5 < s < 1$.

Now let, $T > 0$, $\tau \in [0, T]$, $s > 3/5$, then there exist $j \in [0, n]$, $n \approx T/\Delta T$ and $t \in [0, \Delta T]$ such that

$$u(\tau) = u(t + (j-1)\Delta T) = \overset{j}{v}(t) + \overset{j}{w}(t) = \overset{j}{v}(t) + \overset{j}{z}(t) + U(\tau)w_0, \tag{4.12}$$

thus using interpolation, inequality (2.5), conserved quantities (1.6) and (4.5)-(4.7)

$$\begin{aligned}
\| u(\tau) \|_{\dot{H}^\theta} &\leq \|\overset{j}{v}(t)\|_{\dot{H}^\theta} + \|\overset{j}{z}(t)\|_{\dot{H}^\theta} + \| w_0 \|_{\dot{H}^\theta} \\
&\leq \|\overset{j}{v}(t)\|_{L^2}^{1-\theta} \|\overset{j}{v}(t)\|_{\dot{H}^1}^{\theta} + \|\overset{j}{z}(t)\|_{L^2}^{1-\theta} \|\overset{j}{z}(t)\|_{\dot{H}^1}^{\theta} + \| u_0 \|_{\dot{H}^s} N^{\theta-s} \\
&\leq \|v_j\|_{L^2}^{1-\theta} \|v_j\|_{\dot{H}^1}^{\theta} + \|\overset{j}{z}(t)\|_{L^2}^{1-\theta} \|\overset{j}{z}(t)\|_{\dot{H}^1}^{\theta} + \| u_0 \|_{\dot{H}^s} N^{\theta-s} \\
&\leq 2c_0 N^{\theta(1-s)} + \| u_0 \|_{\dot{H}^s} N^{\theta-s} \\
&\leq c_1 T^{2\theta(1-s)/(5s-3)} + \frac{\| u_0 \|_{\dot{H}^s}}{T^{2(s-\theta)/(5s-3)}},
\end{aligned}$$

where $c_1 = c_1(\| u_0 \|_{H^s})$ and $0 \leq \theta \leq s$.

From inequality (4.12) we get

$$u(\tau) = U(\tau)u_0 + \overset{j}{v}(t) + \overset{j}{z}(t) - U(\tau)v_0,$$

therefore

$$\begin{aligned}
\|\overset{j}{v}(t) + \overset{j}{z}(t) - U(\tau)v_0\|_{\dot{H}^\delta} &\leq 2c_0 N^{\delta(1-s)} + \| u_0 \|_{\dot{H}^s} N^{\delta(1-s)} \\
&\leq c_1 T^{2\delta(1-s)/(5s-3)},
\end{aligned}$$

where $c_1 = c_1(\| u_0 \|_{\dot{H}^s})$ and $s \leq \delta \leq 1$.

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